Journal of Approximation Theory **112**, 73–92 (2001)
doi:10.1006/jath.2001.3594, available online at http://www.idealibrary.com on IDEAL®

Entropy Numbers of Embeddings of Besov Spaces in Generalized Lipschitz Spaces

Fernando Cobos

Departamento de Análisis Matemático, Facultad de Matemáticas, Universidad Complutense de Madrid, E-28040 Madrid, Spain
E-mail: cobos@eucmax.sim.ucm.es

and

Thomas Kühn

Mathematisches Institut, Fakultät für Mathematik und Informatik, Universität Leipzig, Augustusplatz 10/11, D-04109 Leipzig, Germany
E-mail: kuehn@mathematik.uni-leipzig.de

Communicated by Allan Pinkus

Received August 8, 2000; accepted in revised form April 11, 2001

We establish two-sided estimates for entropy numbers of embeddings between certain weighted Banach sequence spaces with mixed norms. These estimates are "almost" sharp, in the sense that upper and lower bounds differ only by logarithmic terms and improve previous results by D. E. Edmunds and D. Haroske (1999, *Dissertationes Math.* **380**, 1–43; 2000, *J. Approx. Theory* **104**, 226–271). As an application we obtain also new upper entropy estimates for embeddings of Besov spaces in generalized Lipschitz spaces.

Key Words: entropy numbers; Besov spaces; Lipschitz spaces; complex interpolation; Gaussian processes.

1. INTRODUCTION

In 1980 Brézis and Wainger [2] showed that every function f in the Sobolev space $H_p^{1+n/p}(\mathbb{R}^n)$, $1 < p < \infty$, satisfies the following Lipschitz type condition

$$|f(x) - f(y)| \leqslant c \, \|x-y\|_2 \, |\log \|x-y\|_2|^{1/p'} \, \|f \mid H_p^{1+n/p}(\mathbb{R}^n)\|$$

for all $x, y \in \mathbb{R}^n$ with $0 < \|x-y\|_2 < 1/2$. Here $\|\cdot\|_2$ denotes the Euclidean norm in $\mathbb{R}^n$, the conjugate index p' of p is given by $1/p + 1/p' = 1$, and the constant c is independent of x, y and f.

0021-9045/01 $35.00

Motivated by this fact, Edmunds and Haroske [5, 6] introduced generalized Lipschitz spaces $\mathrm{Lip}^{(1,-\alpha)}(\mathbb{R}^n)$, $\alpha \geqslant 0$, formed by all complex-valued functions f on $\mathbb{R}^n$ such that the norm

$$\|f \mid \mathrm{Lip}^{(1,-\alpha)}(\mathbb{R}^n)\| = \|f \mid L_\infty(\mathbb{R}^n)\| + \sup_{0<\|x-y\|_2<1/2} \frac{|f(x)-f(y)|}{\|x-y\|_2 \, |\log \|x-y\|_2|^\alpha}$$

is finite. The Brézis–Wainger result says that there is a continuous embedding

$$H_p^{1+n/p}(\mathbb{R}^n) \hookrightarrow \mathrm{Lip}^{(1,-1/p')}(\mathbb{R}^n). \tag{1}$$

This corresponds to the "limiting" case, where the "differential dimension" of both spaces is the same, $s-n/p=1$, of the well-known embeddings

$$H_p^s(\mathbb{R}^n) \hookrightarrow \mathrm{Lip}(\mathbb{R}^n) \qquad \text{if} \quad s > 1+n/p.$$

It was shown in [5] that formula (1) is sharp, in the sense that $H_p^{1+n/p}(\mathbb{R}^n)$ does not embed in $\mathrm{Lip}^{(1,-\alpha)}(\mathbb{R}^n)$ whenever $\alpha < 1/p'$. Moreover, Edmunds and Haroske studied in [5] and [6] embeddings of Triebel–Lizorkin spaces $F^s_{p,q}(\mathbb{R}^n)$ and Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ in Lipschitz type spaces.

When all spaces are defined on a bounded C^∞-smooth domain $\Omega \subseteq \mathbb{R}^n$ then, for appropriate values of the parameters, these embeddings are even compact. In the context of Besov spaces it was shown in [5, Theorem 2.1] that for $0 < p, q \leqslant \infty$ the embedding

$$id_B \colon B^{1+n/p}_{p,q}(\Omega) \to \mathrm{Lip}^{(1,-\alpha)}(\Omega)$$

is compact if and only if $\alpha > \max(1-1/q, 0)$, and two-sided estimates for the entropy numbers of id_B were given in [5, Theorem 3.5] and [6, Theorem 3.11]. The upper entropy estimates were derived from a factorization of id_B through another embedding id of certain weighted sequence spaces. Such factorizations, in turn, were obtained via (sub)atomic decompositions of function spaces as developed by Triebel [25], see also [5].

The aim of the present paper is to improve the known entropy estimates for the above-mentioned embeddings id and id_B. In the sequence space context we obtain almost sharp two-sided estimates, where the gap between upper and lower bounds is at most of logarithmic order. Our methods of proof, both for the upper and for the lower estimates, are quite different from those used in [5, 6]. Our approach to the upper estimates is based on the famous Sudakov minoration principle from probability theory, combined with complex interpolation techniques. These tools are however only available in the framework of Banach spaces, so we cannot treat the quasi-Banach case. In our lower estimates we use direct combinatorial arguments, and the results cover the quasi-Banach case, too.

As an application we derive then new upper entropy estimates for the function space embeddings id_B.

The organization of the paper is as follows. In Section 2 we fix the notation, recall some relevant facts, and establish a few auxiliary results for later use. Section 3 is devoted to entropy estimates for embeddings (and diagonal operators) in sequence spaces, while in Section 4 applications to Besov–Lipschitz embeddings are given.

2. NOTATION AND PRELIMINARIES

a. *Sequence spaces.* Throughout the paper we assume that all Banach spaces under consideration are defined over the field $\mathbb{C}$ of complex numbers, unless explicitly stated otherwise. We use the common notation l_p^n for the space $\mathbb{C}^n$ equipped with the norm

$$\|x\|_p = \begin{cases} \left(\sum_{k=1}^{n} |x_k|^p \right)^{1/p} & \text{for} \quad 1 \leqslant p < \infty \\ \sup_{1 \leqslant k \leqslant n} |x_k| & \text{for} \quad p = \infty. \end{cases}$$

The weighted vector-valued spaces $l_q(w_j\, A_j)$, $1 \leqslant q \leqslant \infty$, where A_j are Banach spaces and the weights $w_j > 0$ are positive real numbers, consist of all sequences $x = (x_j)_{j=1}^{\infty}$ with $x_j \in A_j$ and

$$\|x \mid l_q(w_j\, A_j)\| = \left(\sum_{j=1}^{\infty} w_j^q \, \|x_j \mid A_j\|^q \right)^{1/q} < \infty,$$

with the usual modification of the norm for $q = \infty$. Under this norm, $l_q(w_j\, A_j)$ is clearly a Banach space. By the jth coordinate of $x = (x_j)_{j=1}^{\infty} \in l_q(w_j\, A_j)$ we mean the vector $x_j \in A_j$. We shall work in this paper with the spaces

$$l_q(j^\alpha\, l_p^{M_j}), \qquad \text{where} \quad 1 \leqslant p < \infty, \;\; 1 \leqslant q \leqslant \infty, \;\; M^j \sim 2^{jn} \qquad \text{and} \qquad \alpha \in \mathbb{R}.$$

It sufffices however to consider the model case $M_j = 2^j$, the results and proofs are the same. The symbol $\sim$ has the following meaning. Given two sequences $(a_k)_{k=1}^{\infty}$ and $(b_k)_{k=1}^{\infty}$ of positive real numbers we shall write $a_k \preccurlyeq b_k$, if there is a constant $c > 0$ such that for all $k \in \mathbb{N}$ the inequality $a_k \leqslant c\, b_k$ holds, and $a_k \sim b_k$, if both $a_k \preccurlyeq b_k$ and $b_k \preccurlyeq a_k$ are fulfilled. All constants $c, C, \ldots$ appearing in the sequel are to be understood as positive real numbers which may depend on the involved real parameters $p, q, \alpha, \beta, \ldots \in \mathbb{R}$ but not on integers $j, k, m, n, \ldots \in \mathbb{N}$. Logarithms are always taken to the base 2, $\log = \log_2$.

b. *Entropy numbers.* The kth (dyadic) entropy number of a bounded linear operator $T: X \to Y$ from a Banach space X into another Banach space Y is defined as

$$e_k(T) = \inf\{\varepsilon > 0 : T(B_X) \text{ can be covered by } 2^{k-1} \text{ balls of radius } \varepsilon \text{ in } Y\},$$

where B_X denotes the closed unit ball in X. Entropy numbers are closely related to Kolmogorov's concept of metric entropy, which goes back to the 1930's. Since then it has been successfully applied in many different branches of mathematics. Due to the obvious fact that

$$T \text{ is compact} \qquad \text{if and only if} \quad \lim_{k \to \infty} e_k(T) = 0,$$

the rate of decay of the sequence $(e_k(T))$ can be viewed as a measure for the "degree" of compactness of T. Entropy numbers enjoy many nice properties, for example they are additive and multiplicative, and behave well under interpolation. For these basic properties, more background, and applications to eigenvalue and compactness problems we refer to the monographs by Pietsch [18], König [10], Carl and Stephani [4], Edmunds and Triebel [7], and Triebel [25], and the references given therein, concerning recent applications we mention also the papers by Edmunds and Haroske [5, 6], and by Leopold [15, 16].

For later use we state now some known results; the first one is due to Schütt [21].

LEMMA 1. *Let* $1 \leqslant p \leqslant q \leqslant \infty$, *then*

$$e_k(id: l_p^n \to l_q^n) \sim \begin{cases} 1 & \text{if } \ 1 \leqslant k \leqslant \log 2n \\ \left(\dfrac{\log\left(\dfrac{2n}{k}+1\right)}{k} \right)^{1/p-1/q} & \text{if } \ \log 2n \leqslant k \leqslant 2n \\ 2^{-\frac{k-1}{2n}} n^{1/q-1/p} & \text{if } \ k \geqslant 2n. \end{cases} \tag{2}$$

Originally the asymptotic formula (2), which is essential in many applications, was proved in [21] for *real* spaces, with n on the right hand side instead of $2n$. The *complex* version follows from volume arguments, regarding the n-dimensional complex space l_p^n as $2n$-dimensional real space. For details see [7, Proposition 3.2.2]. In passing we remark that (2) has recently been extended to the quasi-Banach case $0 < p \leqslant q \leqslant \infty$, see [7, 25, 12]. This was again motivated by applications, mainly by the increasing interest during the last decade in quasi-Banach function spaces, for instance in (both classical and generalized) Besov, Sobolev, and Lipschitz spaces.

The following result is due to Gordon, König and Schütt [9, Proposition 1.7]. Formula (3) remains valid for complex spaces, provided the factor $2^{-k/n}$ is replaced by $2^{-k/2n}$.

LEMMA 2. *Let X be a real Banach space with 1-unconditional basis $\{x_j\}_{j=1}^{\infty}$ (for example a symmetric Banach sequence space, or any of the spaces l_p with $1 \leqslant p < \infty$), and let $D_\sigma \colon X \to X$ be the diagonal operator induced by $x_j \mapsto \sigma_j x_j$, where $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant 0$. Then for all $k \in \mathbb{N}$*

$$\sup_{n \in \mathbb{N}} 2^{-k/n}(\sigma_1 \cdots \sigma_n)^{1/n} \leqslant e_{k+1}(D_\sigma) \leqslant 6 \sup_{n \in \mathbb{N}} 2^{-k/n}(\sigma_1 \cdots \sigma_n)^{1/n}. \tag{3}$$

c. *The ℓ-norm.* An important and very useful tool in the theory of Gaussian measures on Banach spaces (and also in Banach space geometry, see e.g. [19]) is the so-called ℓ-norm of an operator $T \colon H \to X$ from a Hilbert space H into a Banach space X. It is defined by

$$\ell(T) = \sup \left(\int_E \|Th\|^2 \, d\gamma_E(h) \right)^{1/2},$$

where the supremum is taken over all finite-dimensional subspaces E of H. The measure γ_E denotes the standard normal distribution on the complex Hilbert space E, regarded as $\mathbb{C}^{\dim E} = \mathbb{R}^{2 \dim E}$. Although Gaussian measures are usually considered on real spaces only, there is no real need for this restriction, and so we do not leave the complex framework.

The following result, which relates the ℓ-norm with entropy numbers, is known as Sudakov's minoration principle, see [22] for the original formulation, [14, Theorem 3.18] for a variant in the language of Gaussian processes, or [11] for the version given below.

LEMMA 3. *There is a constant $c > 0$ such that for all operators T from a Hilbert space H into a Banach space X the inequality*

$$\sup_{k \in \mathbb{N}} k^{1/2} e_k(T^*) \leqslant c \, \ell(T)$$

holds. (Here T^* denotes the dual of T.)

Next we state a consequence of Sudakov's inequality, which will be needed later.

LEMMA 4. *There is a constant $c > 0$ such that the diagonal operator*

$$D: l_1 \to l_\infty, \qquad \text{given by} \quad D(x_n) = ((1+\log n)^{-1} x_n),$$

satisfies for all $k \in \mathbb{N}$ the upper entropy estimate

$$e_k(D) \leqslant c\, k^{-1}.$$

Proof. It is well-known (see e.g. [17, Theorem 9]) that for any diagonal operator $D_\sigma: l_2 \to l_\infty$, generated by $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant 0$, one has

$$\ell(D_\sigma) \sim \sup_{n \in \mathbb{N}} \sigma_n \sqrt{1+\log n}.$$

Taking $\sigma_n = (1+\log n)^{-1/2}$, this yields $\ell(D_\sigma: l_2 \to l_\infty) < \infty$, consequently Lemma 3 implies

$$e_k(D_\sigma: l_1 \to l_2) \preccurlyeq k^{-1/2},$$

and by Tomczak-Jaegermann's results [23] on duality of entropy numbers we obtain

$$e_k(D_\sigma: l_2 \to l_\infty) \preccurlyeq k^{-1/2}.$$

Finally the multiplicativity of the entropy numbers gives the desired estimate for our diagonal operator D

$$e_{2k}(D: l_1 \to l_\infty) \leqslant e_k(D_\sigma: l_1 \to l_2) \cdot e_k(D_\sigma: l_2 \to l_\infty) \preccurlyeq k^{-1}. \quad \blacksquare$$

d. *Complex interpolation.* As already mentioned in the introduction, interpolation techniques are an essential part of our methods of proof. Our general reference to interpolation theory, in particular to the complex method, are the monographs by Bergh and Löfström [1], and Triebel [24]. We will use the same notation and terminology. In particular, if $0 < \theta < 1$ and (A_0, A_1) is an interpolation couple (i.e. a pair of Banach spaces which both embed linearly and continuously into a common Hausdorff vector space), then we denote by $[A_0, A_1]_\theta$ the complex interpolation space. Later on we will need the following result, which is a special case of more general and well-known facts.

LEMMA 5. *Let $1 \leqslant q < \infty$, $\beta \in \mathbb{R}$ and $0 < \theta < 1$. Then one has with equality of norms*

$$[\, l_q(l_\infty^{2^j}), l_q(j^\beta l_1^{2^j})]_\theta = l_q(j^\alpha l_p^{2^j}) \qquad \text{and} \tag{4}$$

$$[c_0(l_\infty^{2^j}), c_0(j^\beta\, l_1^{2^j})]_\theta = c_0(j^\alpha\, l_p^{2^j}), \tag{5}$$

where $1/p = \theta$ and $\alpha = \theta\, \beta$.

Proof. This is a consequence of the interpolation result (i) for vector-valued l_q-spaces, the simple observation (ii), and the well-known basic formula (iii), which are stated below. All three results hold with equality of norms of the involved spaces.

(i) Let (A_j, B_j), $j \in \mathbb{N}$, be any sequence of Banach couples, let $1 \leqslant q < \infty$ and $0 < \theta < 1$. Then

$$[l_q(A_j), l_q(B_j)]_\theta = l_q([A_j, B_j]_\theta).$$

For $q = \infty$ the same is true, if l_q is replaced by c_0, cf. [24, Theorem 1.18.1. and Remarks 1 and 2, p. 122].

(ii) Let A be a Banach space and $\lambda > 0$, and denote by λA the same space equipped with the new norm

$$\|x\|_{\lambda A} := \lambda \|x\|_A.$$

Then by the very definition of the complex method one has

$$[A_0, \lambda A_1]_\theta = \lambda^\theta [A_0, A_1]_\theta.$$

(iii) If $1 \leqslant p_0, p_1 \leqslant \infty$ and $0 < \theta < 1$, then

$$[l_{p_0}, l_{p_1}]_\theta = l_p, \qquad \text{with} \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}. \quad ∎$$

Finally let us recall the nice behaviour of entropy numbers under interpolation, specified to the complex method, which is sufficient for our present purposes. The general result, for arbitrary interpolation functors of type θ, can be found in [18, Theorem 12.1.11]. Here $\mathscr{L}(X, Y)$ stands for the class of all bounded linear operators from X into Y. (A similar result is true when the target space is interpolated, see [18, Theorem 12.1.12].)

LEMMA 6. *Let (A_0, A_1) be an interpolation couple, let B be Banach space, $0 < \theta < 1$ and $m, n \in \mathbb{N}$. Then for every operator $T \in \mathscr{L}(A_i, B)$, $i =$ 0, 1, one has*

$$e_{m+n-1}(T\colon [A_0, A_1]_\theta \to B) \leqslant 2 e_m(T\colon A_0 \to B)^{1-\theta}\, e_n(T\colon A_1 \to B)^\theta.$$

3. ENTROPY ESTIMATES IN SEQUENCE SPACES

First we treat operators in weighted sequence spaces. We begin with a simple, but nevertheless very useful observation. It is obvious that the weighted vector-valued sequence space $l_q\,(w_j\,A_j)$, $1 \leqslant q \leqslant \infty$, (as introduced in Section 2.a) is isometrically isomorphic to to the unweighted space $l_q\,(A_j)$, and clearly the isomorphism is given by the diagonal operator

$$D\colon l_q(w_j\,A_j) \to l_q(A_j), \qquad Dx = (w_j x_j).$$

Moreover, the inverse $D^{-1}\colon l_q(A_j) \to l_q(w_j A_j)$ of D is the diagonal operator generated by the sequence (w_j^{-1}). Therefore, due to the multiplicativity of the entropy numbers, we have

$$\begin{aligned} e_k(id\colon l_q(l_p^{2^j}) \to l_q(j^{-\alpha} l_\infty^{2^j})) &= e_k(id\colon l_q(j^{\alpha} l_p^{2^j}) \to l_q(l_\infty^{2^j})) \\ &= e_k(D_\alpha\colon l_q(l_p^{2^j}) \to l_q(l_\infty^{2^j})), \end{aligned} \tag{6}$$

where the diagonal operator D_α, $\alpha > 0$, is given by

$$D_\alpha(x_j) = (j^{-\alpha} x_j). \tag{7}$$

Roughly speaking this means that weights can be "shifted" from the target space to the first space, or to the operator. This possibility will be extremely useful for the interpolation techniques that we are going to apply in the proof of our main result.

Our first theorem provides upper estimates for the diagonal operators of the form (7).

THEOREM 1. *Let $1 \leqslant p < \infty$, $1 \leqslant q \leqslant \infty$ and $\alpha > 0$. Then the entropy numbers of the diagonal operator*

$$D_\alpha\colon l_q(l_p^{2^j}) \to l_q(l_\infty^{2^j})$$

satisfy the upper estimate

$$e_k(D_\alpha) \preccurlyeq \begin{cases} k^{-1/p}(1+\log k)^{2/p-\alpha} & \text{if } \alpha > 2/p \\ k^{-1/p}(1+\log k)^{1/p} & \text{if } \alpha = 2/p \\ k^{-\alpha/2} & \text{if } \alpha < 2/p. \end{cases} \tag{8}$$

Proof. In order to make the proof more transparent, it will be divided into several steps. Our strategy is as follows. First we deal with the special case $p = 1$. This is the point where in a crucial way Sudakov's minoration comes into play, or more precisely, our auxiliary Lemma 4, which was derived from it. (Note that for $p = 1$ the "critical" value of α is $\alpha = 2/p = 2$.) Applying then complex interpolation (Lemmata 5 and 6) we extend the result to the general case $1 \leqslant p < \infty$.

Step 1. Let $p=1$ and $\alpha>2$, and split α as $\alpha=\beta+\delta+1$ with $\beta>1/q'$ and $\delta>1/q$.

For arbitrary $n\in\mathbb{N}$ consider the projections P_n onto the first n blocks, and Q_n onto the remaining blocks, defined by

$$P_n x=(x_1,\ldots,x_n,0,\ldots) \qquad \text{and} \qquad Q_n x=(0,\ldots,0,x_{n+1},x_{n+2},\ldots).$$

Then we have the following commutative diagram.

$$\begin{array}{ccc} l_q(l_1^{2^j}) & \xrightarrow{D_\alpha Q_n} & l_q(l_\infty^{2^j}) \\ {\scriptstyle D_\beta Q_n}\downarrow & & \uparrow{\scriptstyle D_\delta Q_n} \\ l_1(l_1^{2^j}) & \xrightarrow{D_1} & l_\infty(l_\infty^{2^j}). \end{array}$$

Using Hölder's inequality we obtain the norm estimates

$$\|D_\delta Q_n\|\leqslant\Big(\sum_{j>n} j^{-\delta q}\Big)^{1/q}\sim n^{1/q-\delta} \qquad \text{and similarly}$$

$$\|D_\beta Q_n\|=\|(D_\beta Q_n)^*\|\leqslant\Big(\sum_{j>n} j^{-\beta q'}\Big)^{1/q'}\sim n^{1/q'-\beta},$$

since we have $\delta\, q>1$ and $\beta\, q'>1$.

Note that the operator D_1 is a diagonal operator from l_1 into l_∞ generated by a sequence $\sigma_j\sim(1+\log j)^{-1}$, therefore Lemma 4 implies $e_k(D_1)\preccurlyeq k^{-1}$. The multiplicativity of entropy numbers gives now

$$e_k(D_\alpha Q_n)\leqslant\|D_\beta Q_n\|\, e_k(D_1)\,\|D_\delta Q_n\|\leqslant c\,k^{-1}n^{1/q'-\beta+1/q-\delta}\leqslant c\,k^{-1}n^{2-\alpha}$$

for all $k,n\in\mathbb{N}$ and some constant $c>0$ not depending on n and k.

Our next aim is to estimate the entropy numbers of $D_\alpha P_n$. We recall the well-known inequality (see e.g. Proposition 1.3.1 and formula (1.3.14)' on p. 16 of [4])

$$e_{k+1}(T)\leqslant 4\cdot 2^{-k/2n}\,\|T\|,$$

which is valid for arbitrary operators T of rank n between complex Banach spaces and for all integers $n,k\in\mathbb{N}$.

Since we have

$$\|D_\alpha P_n\|\leqslant\|D_\alpha\|\leqslant 1 \qquad \text{and} \qquad \operatorname{rank}(D_\alpha P_n)=\sum_{j=1}^{n} 2^j<2^{n+1},$$

it follows for $k=8n\cdot 2^n$

$$e_{k+1}(D_\alpha P_n)\leqslant 4\cdot 2^{-8n\cdot 2^n/2\cdot 2^{n+1}}=4\cdot 2^{-2n}.$$

The additivity of entropy numbers yields finally

$$\begin{aligned} e_{2k}(D_\alpha) &\leqslant e_{k+1}(D_\alpha P_n) + e_k(D_\alpha Q_n) \\ &\leqslant 4 \cdot 2^{-2n} + c\, k^{-1} n^{2-\alpha} \sim k^{-1}(1+\log k)^{2-\alpha}, \end{aligned}$$

since clearly $n \sim 1 + \log k$. The proof in our special case is finished.

Step 2. In the case $p=1$, $\alpha=2$ we write again $\alpha = \beta + \delta + 1$, where now $\beta = 1/q'$ and $\delta = 1/q$, and consider a similar commutative diagram as in Step 1, but this time for $D_\alpha P_n$ instead of $D_\alpha Q_n$.

$$\begin{array}{ccc} l_q(l_1^{2^j}) & \xrightarrow{\;D_\alpha P_n\;} & l_q(l_\infty^{2^j}) \\ {\scriptstyle D_\beta P_n}\big\downarrow & & \big\uparrow{\scriptstyle D_\delta P_n} \\ l_1(l_1^{2^j}) & \xrightarrow{\;D_1\;} & l_\infty(l_\infty^{2^j}). \end{array}$$

Since $\beta\, q' = \delta\, q = 1$ we have the norm estimates

$$\|D_\delta P_n\| \leqslant \left(\sum_{j=1}^{n} j^{-\delta q} \right)^{1/q} \sim (1+\log n)^{1/q} \qquad \text{and}$$

$$\|D_\beta P_n\| = \|(D_\beta P_n)^*\| \leqslant \left(\sum_{j=1}^{n} j^{-\beta q'} \right)^{1/q'} \sim (1+\log n)^{1/q'}.$$

Moreover, we have trivially

$$\|D_\alpha Q_n\| = (n+1)^{-\alpha} < n^{-2}.$$

Using again additivity and multiplicativity of entropy numbers we obtain

$$\begin{aligned} e_k(D_\alpha) &\leqslant \|D_\alpha Q_n\| + \|D_\beta P_n\|\; e_k(D_1)\; \|D_\delta P_n\| \\ &\leqslant n^{-2} + c(1+\log n)^{1/q'} \cdot k^{-1} \cdot (1+\log n)^{1/q} \\ &\sim n^{-2} + k^{-1}(1+\log n). \end{aligned}$$

Now let $k \sim n^2$, then clearly $\log k \sim \log n$, and the desired result follows,

$$e_k(D_\alpha) \preccurlyeq k^{-1}(1+\log k).$$

Step 3. If $p=1$, $0<\alpha<2$, we proceed in complete analogy with Step 2. Therefore we only indicate the changes.

Split $\alpha = \beta + \delta + 1$, with $\beta < 1/q'$ and $\delta < 1/q$, where also negative values of β and δ are allowed. Considering the same diagram as in Step 2, we get the norm estimates

$$\|D_\alpha Q_n\| < n^{-\alpha}, \qquad \|D_\delta P_n\| \preccurlyeq n^{1/q-\delta} \qquad \text{and} \qquad \|D_\beta P_n\| \preccurlyeq n^{1/q'-\beta},$$

which gives for all $k, n \in \mathbb{N}$

$$e_k(D_\alpha) \leqslant n^{-\alpha} + c\,k^{-1}\,n^{2-\alpha},$$

and, taking again $k \sim n^2$, we conclude that

$$e_k(D_\alpha) \preccurlyeq n^{-\alpha} \sim k^{-\alpha/2}.$$

Step 4. It remains to prove estimate (8) in full generality, that means for arbitrary $1 \leqslant q \leqslant \infty$, $1 < p < \infty$ and $\alpha > 0$. This will be done by complex interpolation, where the results of the previous steps serve as one "endpoint," while the other "endpoint" will be trivial. Define now $\theta := 1/p$ and $\beta := \alpha\, p$. Then we have clearly $0 < \theta < 1$ and $\alpha = \theta\, \beta$, and by Lemma 5, formula (4), we obtain for $q < \infty$

$$[l_q(l_\infty^{2^j}), l_q(j^\beta l_1^{2^j})]_\theta = l_q(j^\alpha l_p^{2^j}).$$

Now we consider the embeddings (formal identities)

$$id_0 \colon l_q(l_\infty^{2^j}) \to l_q(l_\infty^{2^j})$$
$$id_1 \colon l_q(j^\beta\, l_1^{2^j}) \to l_q(l_\infty^{2^j})$$

The interpolation behaviour of entropy numbers (cf. Lemma 6) yields now for the identity

$$id \colon l_q(j^\alpha\, l_p^{2^j}) \to l_q(l_\infty^{2^j})$$

and arbitrary integers $k \in \mathbb{N}$ the estimate

$$e_k(id) \leqslant 2\, \|id_0\|^{1-\theta}\, \|id_1\|^\theta = 2\, e_k(id_1)^{1/p}.$$

Using now the shifting of weights technique (see (6)), we arrive at the inequality

$$e_k(D_\alpha \colon l_q(l_p^{2^j}) \to l_q(l_\infty^{2^j})) \leqslant e_k(D_\beta \colon l_q(l_1^{2^j}) \to l_q(l_\infty^{2^j}))^{1/p}.$$

Inserting here the results of Steps 1–3 we get the desired estimate (8). In the case $q = \infty$ we use the second interpolation formula (5) of Lemma 5 with c_0 instead of l_q, and get the desired estimates for the operator

$$\widetilde{D_\alpha} : c_0(l_p^{2^j}) \to l_\infty(l_\infty^{2^j}) \qquad \text{instead of}$$
$$D_\alpha \colon l_\infty(l_p^{2^j}) \to l_\infty(l_\infty^{2^j}).$$

Due to the additivity of entropy numbers this implies, for arbitrary integers $n, k \in \mathbb{N}$,

$$e_k(D_\alpha) \leqslant e_k(D_\alpha P_n) + \|D_\alpha Q_n\| \leqslant e_k(\widetilde{D_\alpha} P_n) + n^{-\alpha},$$

and letting $n \to \infty$ this yields

$$e_k(D_\alpha) = e_k(\widetilde{D_\alpha}).$$

The proof is finished. ▮

Now we pass to the lower estimates for the diagonal operators under consideration.

THEOREM 2. *Let $1 \leqslant p < \infty$, $1 \leqslant q \leqslant \infty$ and $\alpha > 0$. Then we have the lower entropy estimate*

$$e_k(D_\alpha : l_q(l_p^{2^j}) \to l_q(l_\infty^{2^j})) \succcurlyeq \begin{cases} k^{-1/p}(1+\log k)^{1/p-\alpha} & \text{if } \alpha > 2/p \\ k^{-\alpha/2} & \text{if } \alpha \leqslant 2/p. \end{cases} \tag{9}$$

Proof. In the case $\alpha > 2/p$ we use Schütt's result [21], stated as Lemma 1. Let $n \in \mathbb{N}$ be given, and let $k \sim 2^{n/2}$, then in particular $n \sim \log k$. Restricting the operator D_α to the n^{th} block and using the multiplicativity of entropy numbers we obtain

$$e_k(D_\alpha) \succcurlyeq n^{-\alpha} e_k(id : l_p^{2^n} \to l_\infty^{2^n}) \sim n^{-\alpha} \left(\frac{\log\left(\frac{2^{n+1}}{k}+1\right)}{k} \right)^{1/p}$$

$$\sim k^{-1/p}(1+\log k)^{1/p-\alpha},$$

and, since $n \in \mathbb{N}$ was arbitrary, the desired inequality is shown.

Now let us turn to the case $\alpha \leqslant 2/p$. Here we use a direct estimate for the entropy numbers, based on combinatorial arguments. For any given integer $n \in \mathbb{N}$ consider the set

$$S := S_1 \times \cdots \times S_{2n+1} \times \{0\} \times \{0\} \times \cdots \subseteq l_q(l_p^{2^j}),$$

where $S_j = \{e_1, ..., e_{2^j}\}$ is the set of unit vectors in $\mathbb{C}^{2^j}$. Then clearly

$$\text{card } S = \prod_{j=1}^{2n+1} 2^j = 2^{(2n+1)(n+1)}.$$

Let h denote the Hamming distance on S, defined by

$$h(x, y) := \text{card}\{j \in \{1, ..., 2n+1\} : x_j \neq y_j\}.$$

For every fixed element $x \in S$ and every integer m with $1 \leqslant m \leqslant 2n+1$ we have the estimate

$$\operatorname{card}\{y \in S : h(x, y) \leqslant m\} \leqslant 2^{(2n+1)m}.$$

(Indeed, all elements $y \in S$ with $h(x, y) \leqslant m$ can be found as follows. First choose an arbitrary subset $J \subseteq \{1, ..., 2n+1\}$ of cardinality card $J = m$. There are of course $\binom{2n+1}{m} \leqslant 2^{2n+1}$ possible choices. For the coordinates $j \notin J$ define $y_j := x_j$, and for $j \in J$ take $y_j \in S_j$ arbitrarily. There are $2^j \leqslant 2^{2n+1}$ such ways in each of the m coordinates, altogether $2^{(2n+1)m}$.)

Let now A be any subset of S of cardinality card $A \leqslant 2^{n^2}$ and take $m := [n/2]$. Then we have

$$\begin{aligned}\operatorname{card}\{y \in S : \exists x \in A \text{ with } h(x, y) \leqslant m\} &\\ \leqslant \sum_{x \in A} \operatorname{card}\{y \in S : h(x, y) \leqslant m\} &\\ \leqslant \operatorname{card} A \cdot 2^{(2n+1)m} \leqslant 2^{n^2+(2n+1)n/2} < \operatorname{card} S,\end{aligned}$$

therefore there exists an element $y \in S$ with $h(x, y) \geqslant [n/2]+1 \geqslant n/2$ for all $x \in A$. So we can find inductively a subset $A \subseteq S$ of cardinality card $A \geqslant 2^{n^2}$ such that $h(x, y) \geqslant n/2$ for any pair of distinct elements $x, y \in A$. This implies

$$\begin{aligned}\|D_\alpha x - D_\alpha y\|_{l_q(l_\infty^{2^j})} &= \left(\sum_{j=1}^{2n+1} j^{-\alpha q} \|x_j - y_j\|_\infty^q\right)^{1/q}\\ &\geqslant (2n+1)^{-\alpha} \left(\sum_{j=1}^{2n+1} \|x_j - y_j\|_\infty^q\right)^{1/q} \geqslant (2n+1)^{-\alpha} \left(\frac{n}{2}\right)^{1/q},\end{aligned}$$

where we used that $\|x_j - y_j\|_\infty = 1$, if $x_j \neq y_j$ and that there are $h(x, y)$ coordinates j with $x_j \neq y_j$. Taking into account that the set $(2n+1)^{-1/q} A$ is contained in the unit ball of $l_q(l_p^{2^j})$ it follows

$$e_{n^2}(D_\alpha) \geqslant \frac{1}{2} \cdot (2n+1)^{-1/q-\alpha} \cdot \left(\frac{n}{2}\right)^{1/q} \sim n^{-\alpha},$$

whence, since $n \in \mathbb{N}$ was arbitrary, we arrive at the desired inequality

$$e_k(D_\alpha) \succcurlyeq k^{-\alpha/2}. \quad \blacksquare$$

The proof shows that the lower estimate (9) holds even in the quasi-Banach case $0 < p < \infty$, $0 < q \leqslant \infty$. It seems very likely that also the upper estimate (8) is valid in the quasi-Banach case. To prove this would require however new techniques.

In the Banach case $1 \leqslant p < \infty$, $1 \leqslant q \leqslant \infty$, the upper and lower bounds, given in (8) and (9), coincide for $0 < \alpha < 2/p$, while for $\alpha \geqslant 2/p$ there remains only a small gap of logarithmic order $(\log k)^{1/p}$ between the two bounds. In general we do not know the *exact* asymptotic behaviour, but we conjecture that the upper bound is sharp. This conjecture is supported by the following result in the case $q = \infty$.

Note added in proof. The conjecture was recently confirmed by E. Belinsky (Entropy numbers of vector-valued diagonal operators, preprint, University of the West Indies, 2001). He showed that estimate (8) gives the exact behaviour, even in the quasi-Banach setting.

PROPOSITION 1. *If $1 \leqslant p < \infty$ and $\alpha > 2/p$, then*

$$e_k(D_\alpha : l_\infty(l_p^{2^j}) \to l_\infty(l_\infty^{2^j})) \sim k^{-1/p}(\log k)^{2/p-\alpha}.$$

Proof. The upper estimate is covered by Theorem 1, so we only have to prove the lower one. First we consider the special case $p = 1$, where we proceed as in [21], using combinatorial arguments. By [20, Aufgabe I.29] we have for all $k, m \in \mathbb{N}$

$$\operatorname{card}\left\{x \in \mathbb{Z}^m : \sum_{j=1}^{m} |x_j| \leqslant k\right\} = \sum_{j=0}^{k \wedge m} 2^j \binom{m}{j}\binom{k}{j},$$

where we denoted $k \wedge m = \min(k, m)$. In the case $k \leqslant m$ it follows that

$$\sum_{j=0}^{k \wedge m} 2^j \binom{m}{j}\binom{k}{j} \geqslant 2^k \binom{m}{k} \geqslant 2^k \prod_{i=0}^{k-1} \frac{m-i}{k-i} \geqslant \left(\frac{2m}{k}\right)^k.$$

Given any $n \in \mathbb{N}$, we consider now the subset $A \subseteq l_\infty(l_1^{2^j})$,

$$A = \{x = (x_j)_{j=1}^{\infty} : x_j \in \mathbb{Z}^{2^j}, \|x_j\|_1 \leqslant 2^{n+1} \text{ for } n < j \leqslant 2n,$$
$$\text{and } x_j = 0 \text{ for all other } j \in \mathbb{N}\},$$

and estimate its cardinality by

$$\operatorname{card} A = \prod_{j=n+1}^{2n} \operatorname{card}\{x \in \mathbb{Z}^{2^j} : \|x\|_1 \leqslant 2^{n+1}\} \geqslant \left(\prod_{j=n+1}^{2n} 2^{j-n}\right)^{2^{n+1}}.$$

This implies

$$\log \operatorname{card} A \geqslant 2^{n+1} \sum_{j=1}^{n} j \geqslant 2^n n^2.$$

Clearly, the set $B := 2^{-(n+1)}A$ is contained in the unit ball of $l_\infty(l_1^{2^j})$, and for any two distinct elements $x, y \in B$ the inequality

$$\|D_\alpha x - D_\alpha y\|_\infty \geqslant 2^{-(n+1)}(2n)^{-\alpha}$$

holds. Now let $k = 2^n n^2$, whence $\log k \sim n$. It follows

$$e_k(D_\alpha) \geqslant \tfrac{1}{2}\cdot 2^{-n}(2n)^{-\alpha} = 2^{-(2+\alpha)}k^{-1}n^{2-\alpha} \sim k^{-1}(\log k)^{2-\alpha}.$$

Since the integer $n \in \mathbb{N}$ was arbitrary, this proves the desired assertion

$$e_k(D_\alpha) \succcurlyeq k^{-1}(\log k)^{2-\alpha}.$$

Now we pass to the general case $1 < p < \infty$, $\alpha > 2/p$. First we note that by complex interpolation (for details we refer to [13]) between diagonal operators of the form $D_\alpha: l_\infty(l_1^{2^j}) \to l_\infty(l_\infty^{2^j})$ and the identity in $l_\infty(l_1^{2^j})$ one can derive from the upper estimate (8) the result

$$e_k(D_\beta: l_\infty(l_1^{2^j}) \to l_\infty(l_p^{2^j})) \preccurlyeq k^{-1/p'}(\log k)^{2/p'-\beta},$$

valid for all $\beta > 2/p'$. Fixing any such β, we have $\alpha+\beta > 2$, and by multiplicativity of entropy numbers we obtain

$$\begin{aligned} k^{-1}(\log k)^{2-\alpha-\beta} &\preccurlyeq e_{2k}(D_{\alpha+\beta}: l_\infty(l_1^{2^j}) \to l_\infty(l_\infty^{2^j})) \\ &\leqslant 2e_k(D_\beta: l_\infty(l_1^{2^j}) \to l_\infty(l_p^{2^j}))\, e_k(D_\alpha: l_\infty(l_p^{2^j}) \to l_\infty(l_\infty^{2^j})) \\ &\preccurlyeq k^{-1/p'}(\log k)^{2/p'-\beta}\, e_k(D_\alpha: l_\infty(l_p^{2^j}) \to l_\infty(l_\infty^{2^j})), \end{aligned}$$

where we used in the first inequality the result in the special case. It follows

$$e_k(D_\alpha: l_\infty(l_p^{2^j}) \to l_\infty(l_\infty^{2^j})) \succcurlyeq k^{-1/p}(\log k)^{2/p-\alpha}. \quad \blacksquare$$

4. APPLICATIONS

Finally we apply the results of the preceding section to embeddings of Besov spaces in spaces of Lipschitz type.

THEOREM 3. *Let* $1 \leqslant p < \infty$, $1 \leqslant q \leqslant \infty$, $\alpha > 1/q'$, *set* $\alpha_* = 2/p + 1/q'$, *and let* $\Omega \subseteq \mathbb{R}^n$ *be any bounded domain with* C^∞*-smooth boundary (for example the open Euclidean unit ball). Then the entropy numbers of the embedding*

$$id_B: B^{1+n/p}_{p,q}(\Omega) \to Lip^{(1,-\alpha)}(\Omega)$$

satisfy the upper estimate

$$e_k(id_B) \preccurlyeq \begin{cases} k^{-1/p}(1+\log k)^{\alpha_*-\alpha} & \textit{if} \quad \alpha > \alpha_* \\ k^{-1/p}(1+\log k)^{1/p} & \textit{if} \quad \alpha = \alpha_* \\ k^{-\frac{\alpha-1/q'}{2}} & \textit{if} \quad \alpha < \alpha_*. \end{cases} \tag{10}$$

Proof. The technique of the proof is the same as in [6], therefore we shall only sketch the main steps. The whole proof relies essentially on a factorization of id_B through an embedding of certain sequence spaces, and the multiplicativity of entropy numbers. This factorization is based on subatomic decompositions of functions $f \in B^s_{p,q}(\mathbb{R}^n)$, as developed by Triebel. For the details of this construction we refer to [25, Chapter III.14], here we only explain the relevant ideas in our special situation $s = 1 + n/p$.

Consider the cube $Q = [-1/2, 1/2]^n$ and choose $\psi \in \mathcal{S}(\mathbb{R}^n)$, in the Schwartz space of complex-valued, rapidly decreasing, infinitely differentiable functions on $\mathbb{R}^n$, with support

$$\operatorname{supp} \psi \subseteq [-r/2, r/2]^n \qquad \text{for some real number} \quad r > 1$$

$$\text{and} \qquad \sum_{m \in \mathbb{Z}^n} \psi(x-m) = 1 \qquad \text{for all} \quad x \in \mathbb{R}^n.$$

Given an integer $j \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and a multiindex $\beta \in \mathbb{N}_0^n$ let $\psi_\beta(x) = x^\beta \psi(x)$. The building blocks, called "quarks" in [25], of the above-mentioned decomposition are the functions $\psi_{\beta, j, m}$, defined by

$$\psi_{\beta, j, m}(x) := 2^{-j} \psi_\beta(2^j x - m), \qquad x \in \mathbb{R}^n.$$

Then one can assign, in a *linear way*, to every function $f \in B^{1+n/p}_{p,q}(\mathbb{R}^n)$ a sequence of complex numbers $\lambda = (\lambda_{\beta, j, m})$, indexed by $\beta \in \mathbb{N}_0^n$, $j \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$, such that f can be represented as a series of the form

$$f = \sum_{\beta \in \mathbb{N}_0^n} \sum_{j \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda_{\beta, j, m} \psi_{\beta, j, m}, \tag{11}$$

convergence being in the space $\mathcal{S}'(\mathbb{R}^n)$ of tempered distributions, such that

$$\|f \mid B^{1+n/p}_{p,q}(\mathbb{R}^n)\| \sim \|\lambda \mid l_\infty(2^{\delta |\beta|} l_q(l_p))\|$$
$$:= \sup_{\beta \in \mathbb{N}_0^n} 2^{\delta |\beta|} \left(\sum_{j=0}^{\infty} \left(\sum_{m \in \mathbb{Z}^n} |\lambda_{\beta, j, m}|^p \right)^{p/q} \right)^{1/q},$$

with the usual modification for $q=\infty$. Here $\delta>0$ is some sufficiently large real number, which is independent of f, and also the equivalence constants do not depend on f.

If f has compact support, say $\operatorname{supp} f \subseteq \Omega$ for some C^∞-smooth domain, then the inner sum over m extends in fact only over a finite number of nonzero summands, since the coefficients vanish for all those "quarks" $\psi_{\beta,j,m}$, whose supports are disjoint from Ω. This number, call it M_j, depends only on the level j, but not on the multiindex $\beta \in \mathbb{N}_0^n$, moreover it can be controlled by $M_j \leqslant c\, 2^{jn}$, where the constant c is determined by the size of Ω and the parameter $r>1$, which describes the possible overlap of the supports of the "quarks" of a given level j. Applying the standard extension-restriction procedure for Besov spaces over domains in $\mathbb{R}^n$ we obtain a bounded operator

$$S: B^{1+n/p}_{p,q}(\Omega) \to l_\infty(2^{\delta|\beta|}\, l_q(l_p^{M_j})),$$

assigning to every function $f \in B^{1+n/p}_{p,q}(\Omega)$ the family $\lambda=(\lambda_{\beta,j,m})$ of all nonzero coefficients which appear in the representation (11) of f. On the other hand, one can show similarly as in [5], see the computations on pages 27–28, that the linear operator

$$T: l_\infty(l_q(\langle j\rangle^{-\alpha+1/q'}\, l_p^{M_j}) \to \mathrm{Lip}^{(1,-\alpha)}(\Omega), \qquad \text{defined by}$$

$$T\lambda = \sum_{\beta\in\mathbb{N}_0^n} \sum_{j\in\mathbb{N}_0} \sum_{m=1}^{M_j} \lambda_{\beta,j,m}\, \psi_{\beta,j,m},$$

is bounded. (Here we used the same notation $\langle j\rangle = 1+j$ as in [5, 6].) Therefore we have the commutative diagram

$$\begin{array}{ccc} B^{1+n/p}_{p,q}(\Omega) & \xrightarrow{\;id_B\;} & \mathrm{Lip}^{(1,-\alpha)}(\Omega) \\ S\downarrow & & \uparrow T \\ l_\infty(2^{\delta|\beta|} l_q(l_p^{M_j})) & \xrightarrow{\;id\;} & l_\infty(l_q(\langle j\rangle^{-\alpha+1/q'}\, l_\infty^{M_j})), \end{array}$$

where the l_∞-spaces are defined over the index set $\mathbb{N}_0^n$. The restriction id_β of the embedding id in the diagram to the coordinate $\beta\in\mathbb{N}_0^n$ of the vector-valued l_∞-spaces is equal to

$$2^{\delta|\beta|}\cdot id: l_q(l_p^{M_j}) \to l_q\,(\,\langle j\rangle^{-\alpha+1/q'}\, l_\infty^{M_j}),$$

thus the entropy numbers of id_β can be controlled by the estimates of Theorem 1. In order to obtain entropy estimates for the embedding id one can argue similarly as in [25, Theorem 9.2]: Using elementary properties of entropy numbers one can show that if

$$u\colon A \to B$$

is a bounded linear operator between two Banach spaces with

$$e_k(u) \preccurlyeq k^{-\alpha}(1+\log k)^{\beta} \qquad \text{for some} \quad \alpha > 0 \text{ and } \beta \in \mathbb{R},$$

then the operator

$$\tilde{u}\colon l_\infty(2^{\delta j} A) \to l_\infty(B), \qquad \tilde{u}x = (ux, ux, \ldots)$$

satisfies the same entropy estimate, for every given $\delta > 0$. Applying this general procedure in our special situation we can carry over the entropy estimates of the id_β's to the embedding id (in the diagram), and by the multiplicativity of entropy numbers finally to the embedding id_B. ∎

Finally we compare our results for the function space embeddings id_B with those of Edmunds and Haroske, who treated also the quasi-Banach case. In the context of Banach spaces their estimates [5, Theorem 4.10] and [6, Theorem 3.11] read as follows.

If $1 \leqslant p < \infty$, $1 \leqslant q \leqslant \infty$ and $\alpha > 1/q'$, then

$$k^{-1/p}(1+\log k)^{-\alpha} \preccurlyeq e_k(id_B) \preccurlyeq \begin{cases} k^{-1/p}(1+\log k)^{\alpha^*-\alpha} & \text{if} \quad \alpha > \alpha^* \\ k^{-1/p}(1+\log k)^{1+2/p} & \text{if} \quad \alpha = \alpha^* \\ k^{-\frac{\alpha-1/q'}{2+p}} & \text{if} \quad \alpha < \alpha^*, \end{cases} \tag{12}$$

where the "critical index" is given by $\alpha^* = 1+2/p+1/q'$. Since $\alpha^* > \alpha_*$, our estimate (10) improves (12) in several respects.

First, we extended the range, where the upper estimate is exact up to log-terms, from $\alpha > \alpha^*$ to $\alpha > \alpha_*$. Moreover we obtained a better upper bound, since in (10) the exponent of the logarithm is smaller than in (12).

Second, for $\alpha < \alpha^*$ we got even an improvement in the exponent of the power term in the upper bound. (Note that our exponent $\frac{\alpha-1/q'}{2}$ is independent of p, while for the exponent in (12) $\lim_{p\to\infty} \frac{\alpha-1/q'}{2+p} = 0$.)

It seems very likely that the estimate (10) is sharp up to log-terms, at least in the Banach space case. However, for proving (or disproving) this conjecture, as well as for handling the quasi-Banach case, another discretization method is needed.

ACKNOWLEDGMENTS

We thank J. Creutzig (Jena) for bringing to our attention the preprint of Gao's paper [8], and D. E. Edmunds (Sussex), and D. D. Haroske (Jena) for sending us a preliminary version of their article [6], which motivated the present paper. The first named author has been supported in part by DGES (PB97-0254) and the second named author gratefully acknowledges the hospitality of the Departamento de Análisis Matemático at Universidad Complutense de Madrid, where a first draft of this paper was written.

REFERENCES

1. J. Bergh and J. Löfström, "Interpolation Spaces," Springer-Verlag, Berlin/Heidelberg/New York, 1976.
2. H. Brézis and S. Wainger, A note on limiting cases of Sobolev embeddings and convolution inequalities, *Comm. Partial Differential Equations* **5** (1980), 773–789.
3. B. Carl, Entropy numbers of diagonal operators with an application to eigenvalue problems, *J. Approx. Theory* **32** (1981), 135–150.
4. B. Carl and I. Stephani, "Entropy, Compactness and Approximation of Operators," Cambridge Univ. Press, Cambridge, UK, 1990.
5. D. E. Edmunds and D. Haroske, Spaces of Lipschitz type, embeddings and entropy numbers, *Dissertationes Math.* **380** (1999), 1–43.
6. D. E. Edmunds and D. D. Haroske, Embeddings in spaces of Lipschitz type, entropy and approximation numbers, and applications, *J. Approx. Theory* **104** (2000), 226–271.
7. D. E. Edmunds and H. Triebel, "Function Spaces, Entropy Numbers and Differential Operators," Cambridge Univ. Press, Cambridge, UK, 1996.
8. F. Gao, Metric entropy of convex hulls, *Israel J. Math.* (2001), to appear.
9. Y. Gordon, H. König, and C. Schütt, Geometric and probabilistic estimates for entropy and approximation numbers of operators, *J. Approx. Theory* **49** (1987), 219–239.
10. H. König, "Eigenvalue Distribution of Compact Operators," Birkhäuser, Basel, 1986.
11. T. Kühn, γ-Radonifying operators and entropy ideals, *Math. Nachr.* **107** (1982), 53–58.
12. T. Kühn, A lower estimate for entropy numbers, *J. Approx. Theory* **110** (2001), 120–124.
13. T. Kühn and T. Schonbek, Entropy numbers of diagonal operators between vector-valued sequence spaces, *J. London Math. Soc.*, to appear.
14. M. Ledoux and M. Talagrand, "Probability in Banach Spaces," Springer-Verlag, Berlin/Heidelberg/New York, 1991.
15. H.-G. Leopold, Embeddings and entropy numbers in Besov spaces of generalized smoothness, *in* "Proc. Conf. Function Spaces V, Poznan," 1998.
16. H.-G. Leopold, Embeddings for general weighted sequence spaces and entropy numbers, *in* "Proc. Conf. Function Spaces, Differential Operators and Nonlinear Analysis, Syöte, 1999," pp. 170–186, Math. Inst. Czech Republic, Prague, 2000.
17. W. Linde and A. Pietsch, Mappings of Gaussian cylindrical measures in Banach spaces, *Teor. Verojatnost. i Primenen.* **19** (1974), 472–487. [Russian; English translation in *Theoret. Probab. Appl.* **19** (1974), 445–460.]
18. A. Pietsch, "Operator Ideals," VEB Deutscher Verlag der Wiss. Berlin, 1978, and North-Holland, Amsterdam/New York, 1980.
19. G. Pisier, "The Volume of Convex Bodies and Banach Space Geometry," Cambridge Univ. Press, Cambridge, UK, 1989.

20. G. Pólya and G. Szegö, "Aufgaben und Lehrsätze aus der Analysis," Erster Band, Springer-Verlag, Berlin/New York, 1954.
21. C. Schütt, Entropy numbers of diagonal operators between symmetric Banach spaces, *J. Approx. Theory* **40** (1984), 121–128.
22. V. N. Sudakov, Gaussian random processes, and measures of solid angles in Hilbert space, *Dokl. Akad. Nauk SSSR* **197** (1971), 43–45. [Russian; English transl. in *Soviet Math. Dokl.* **12** (1971), 412–415.]
23. N. Tomczak-Jaegermann, Dualité des nombres d'entropie pour des opérateurs à valeurs dans un espace de Hilbert, *C. R. Acad. Sci. Paris, Ser. I* **305** (1987), 299–301.
24. H. Triebel, "Interpolation Theory, Function Spaces, Differential Operators," North-Holland, Amsterdam/New York/Oxford, 1978.
25. H. Triebel, "Fractals and Spectra," Birkhäuser, Basel, 1997.